ABSTRACTS

ON SOME GENERALIZATIONS OF FLOWS

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We define here a flow with limited intersection of its worldlines. Then we construct and solve some functional equations for such flow using some kind of set embedding. We demonstrate our ideas by known examples studied in the past by different authors. Finally, we also accent the connection to higher order ordinary differential equations.

SOME NEW RESULTS ON THE DYNAMICS OF THE THUE-MORSE SYSTEM OF DIFFERENCE EQUATIONS

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The Thue-Morse system of difference equations was introduced in [1] as a model to understand the electric behavior (conductor or insulator) of an array of electrical punctual positive charges occupying positions following a one dimensional distribution of points called a Thue-Morse chain. Unfolding the system, we obtain the two-dimensional dynamical system in the plane given by

$$F(x; y) = (x(4 - x - y); xy).$$

The interest of such system was stated by A.N. Sharkovsky as an open problem. The most interesting dynamics of the system is developed inside an invariant plane triangle, where hyperbolic periodic points of almost all period appear, there are subsets of transitivity and invariant curves of spiral form around the unique inside fixed point.

In this talk we will present some results concerning the behavior of all points outside the triangle, completing the known dynamics of the system. In fact we have obtained that outside the triangle, there are periodic points of almost all period, most of points eventually move to infinite and there exists also a non-wandering set Δ where $F|\Delta$ is topologically conjugate to the shift on two symbols. Such new results has an interesting interpretation in terms of the physics of the physical problem. We will also present a way of doing a graphical visualization of the dynamics of the system.

[1] Y. Avishai, D. Berend, *Transmission through a Thue-Morse chain*, Physical Review B 45.6 (2011), 2717-2724.

SHARKOVSKY PROBLEMS OF ONE TRACE MAP

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We consider the trace map

$$F(x,\, y) = (xy,\, (x-2)^2), \tag{1}$$

which arises in quasicrystal phisics (see [1]).
The problems for studying of some dynamical aspects of the map F was formulated by A. N. Sharkovsky in [2]. We formulate here the complete result concerning the nonwandering set $\Omega(F)$ of map (1). For this goal we distinguish the following sets on the plane xOy:
(1) the closed triangle $\Delta = \{(x;y) \in \mathbb{R}^2 : x,\, y \geqslant 0, x+y \leqslant 4\}$;
(2) the unbounded set $G_\Delta = \{(x;y) \in \mathbb{R}^2 : x,\, y \geqslant 0, x+y \geqslant 4\}$;
(3) the unbounded set $D_\infty = \{(x;y) \in \mathbb{R}^2 : x \geqslant 3, y \geqslant 1\}$;
(4) the unbounded set $\widetilde{G} = G_\Delta \cap (\bigcup\limits_{i=0}^{+\infty} F^{-i}(D_{+\infty})$, where $F^{-i}(D_\infty)$ is the i-th complete preimage of the set D_∞;
(5) the unbounded set $G' = G_\Delta \setminus int\widetilde{G}$, where $int(\cdot)$ is the interior of a set;
Theorem *The nonwandering set* $\Omega(F)$ *of map* (1) *is the union of the triangle* Δ *and perfect nowhere dense in* G_Δ *set* G' *satisfying*
(1) *the set* G' *is the union of unbounded curves such that* $G' \cap G^\circ_\Delta$ *is* F*-completely invariant local lamination of codimension 1 in the set* G°_Δ*, in addition, the set of algebraic curves is everywhere dense in* G'*;*
(2) *the map* $F_{|\Delta}$ *is topologically mixing, and its periodic points are everywhere dense in* Δ*.*

Let us note that claim (2) of Theorem gives the positive answer on two problems by A.N. Sharkovsky:

1. Is restriction of map (1) on Δ topologically transitive?
2. Are periodic points of F everywhere dense in Δ?

This is the joint work with L. S. Efremova.

[1] Y. Avishai, D. Berend, *Transmission through a Thue-Morse chain*, Phys. Rev. B. **45** (1992), 2717—2724.

[2] A.N. Sharkovsky, Problem list, Int. Conf. "Low Dimensional Dynamics" (Oberwolfach, Germany, April 25 – May 1 1993), Tagungsbericht, 17, (1993).

[3] S.S. Bel'mesova, L.S. Efremova, *On the concept of the integrability for discrete dynamics systems: criterion, applications to the mathematical problems of quasicrystal physics*, Proceedings of NOMA'13. Workshop. Zaragoza, Spain. September 3 – 4, 2013. p. 13–14.

A COMPOSITE FUNCTIONAL EQUATION RELATED TO CONJUGATION OF SOME FAMILIES OF TRANSFORMATIONS

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We deal with the following problem closely related to the problem of determination of the von Neumann-Morgenstern utility functions invariant with respect to some classes of transformations. Assume that $T \neq \emptyset$, $g, h : \mathbb{R} \to \mathbb{R}$ are continuous bijections, $k, K : T \to \mathbb{R} \setminus \{0\}$ and $l, L : T \to \mathbb{R}$. For every $t \in T$, let $g_t, h_t : \mathbb{R} \to \mathbb{R}$ be given by

$$g_t(x) = g^{-1}(k(t)g(x) + l(t)) \quad \text{for} \quad x \in \mathbb{R}$$

and

$$h_t(x) = h^{-1}(K(t)h(x) + L(t)) \quad \text{for} \quad x \in \mathbb{R}.$$

Determine all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f \circ g_t = h_t \circ f \quad \text{for} \quad t \in T.$$

The problem leads to a composite functional equation. We determine the solutions of the equation and present some applications of our results.

DISTRIBUTIONAL CHAOS VIA SEMICONJUGACY

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Let (Y, g) be a distributionally chaotic system. Assume that (X, f) is another topological dynamical system semiconjugated to (X, f) via continuous surjective map $\varphi : X \to Y$. This means that $\varphi \circ f = g \circ \varphi$. The talk will be devoted to the following open question: *Does the existence of distributionally chaotic factor (Y, g) imply the same chaos for the system (X, f)?*

UNITARY SEMI-STOCHASTIC MATRICES

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Gdańsk University of Technology, Gdańsk, Poland

A matrix of complex numbers is said to be *semi-stochastic* if for every column the sum of elements equals one. With the use of unitary semi-stochastic matrices a multi-instant quantum pure states are introduced. For two instants, say 0 and 1, a pure state at instant 0 represented by a unit vector $\Psi^{(0)} = [\psi_1^{(0)}, \psi_2^{(0)}, \ldots, \psi_d^{(0)}]$ and the unitary semi-stochastic matrix $U^{(1|0)} = [u_{j,k}^{(1|0)};\ (j,k) \in \{1,2,\ldots,d\}^2\,]$, the two-instant state is defined as the (unit) d^2-dimensional vector

$$\Psi^{(1,0)} = \left[\psi_{j,k}^{(1,0)} := u_{j,k}^{(1|0)} \cdot \psi_k^{(0)};\ (j,k) \in \{1,2,\ldots,d\}^2\,\right].$$

Let us note, that the state at instant 1 is given by

$$\Psi^{(1)} = \left[\psi^{(1)} := \sum_k \psi_{j,k}^{(1,0)}\ ;\ j = 1,2,\ldots,d\right],$$

whereas, due to semi-stochastic property, $\psi_k^{(0)} = \sum_j \psi_{j,k}^{(1,0)}$, $k = 1,2,\ldots,d$. The *reverse* two-instant state is defined by

$$\Psi^{(0,1)} = \left[\psi_{k,j}^{(0,1)} := u_{k,j}^{(0|1)} \cdot \psi_j^{(1)};\ (k,j) \in \{1,2,\ldots,d\}^2\,\right],$$

where $u_{k,j}^{(0|1)} = \overline{u_{j,k}^{(1|0)}}$ form the inverse unitary matrix. The main goal is to present a partial solution of the problem of reversibility defined as follows: Does there exist a complex number α of modulus 1, that $\psi_{k,j}^{(0,1)} = \alpha\psi_{j,k}^{(1,0)}$ for all pairs (j,k)?

FROM THE CONCEPT OF INTEGRABILITY TO SKEW PRODUCTS AND TRACE MAPS

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The concept of integrability suggested by R.I. Grigorchuk for a polynomial discrete dynamical system in [1] is generalized for an arbitrary discrete dynamical system in the plane (see [2]).
With the use of introduced definition the criterion of integrability for a discrete dynamical system is proved. This criterion makes it possible to reduce an integrable dynamical system to a dynamical system of the skew products class.
These results and developed technique for skew products are applied for investigation of the trace map $F(x, y) = (xy, (x-2)^2)$, which arises in quasicrystal physics.
Geometric constructions of Denjoy type are described, which are base for the complete description of the structure of the nonwandering set of the above map [3].
A.N. Sharkovsky's problems for the exterior of the invariant triangle are solved.

It is a joint work with S.S. Bel'mesova.

[1] R.I. Grigorchuk, A. Žuk, *The Lamplighter group as a group generated by a 2-state automata, and its spectrum*, Geometriae Dedicata **87** (2001), 209—244.

[2] S.S. Bel'mesova, L.S. Efremova, *On the concept of integrability for discrete dynamical systems: criterion, application to the mathematical problems of quasicrystal physics*, Proceedings of NOMA'13: International Workshop on Nonlinear Maps and their Applications. Zaragoza, Spain, 3rd-4th of September, 2013, (2013), 13–14.

[3] S.S. Bel'mesova, L.S. Efremova, *On the concept of integrability for discrete dynamical systems. Investigation of wandering points of some trace map*, In: Nonlinear Maps and Their Applications, Springer Proceedings in Mathematics and Statistics, Springer, New York, 2015 (to appear).

ON THE SINGULARITIES OF THE UNKNOWNS OF A TWO-PLACE FUNCTIONAL EQUATION ARISING FROM A NETWORK MODEL

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Recently, a certain class of two-place functional equations has applications in communication, and in wireless three-hope networks. This particular class stems naturally from modeling two-queue queueing systems. It takes the general form

$$P(x,y) = \frac{A(x,y)P(x,0) + B(x,y)P(0,y) + C(x,y)P(0,0) + D(x,y)}{H(x,y)}, \tag{1}$$

where the functions A, B, C, D, and H are known polynomials of different degrees in two complex variables x and y. There are no general solution methodology available to (1). In this talk I will introduce a methodology to compute the potential singularities of the unknowns of a special case of (1). This special equation arises from a network gateway modelled as two-back-to-back interfering queues.

This is a joint work with Wolfgang Förg-Rob

APPLICATION OF NONLINEAR DYNAMICS TO CHAOTIC PRNG DESIGN

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Chaotic dynamics application to robust pseudo random number generators (PRNG) design encounters a growing interest thanks to common required properties for message encryption: both cryptography and non-linear dynamics perform the non-linear message transformation. The computer application of the non-linear systems exhibits the deterministic transformation $x_{n+1} = f(x_n)$ but the generated sequences look like real random and unpredictable.

Not all chaotic systems could be applied for cryptography. The non-linear system should demonstrate strong chaoticity measured by largest Lyapunov exponent, implying reactive transient period while slight changes in initial conditions were made and should successfully pass randomness tests: NIST, auto-correlation, cross-correlation, uniform distribution [1]. Consequently chaotic PRNG design is a challenging task.

The purpose of our work is to design robust chaotic generator using non-linear dynamics. The idea is based on combining dynamically two different maps: logistic $f_\mu \equiv L_\mu = 1 - \mu x^2$ and tent $f_\mu \equiv T_\mu = 1 - \mu|x|$ with state complexity, ring and auto coupling [2]. The achieved chaotic PRNG demonstrates strong chaoticity and excellent randomness properties. All tests have been successfully passed promising robust application to real life cryptosystems. The particular study was devoted to the system topologically mixing; to the invariant measure of the each state and between the states. Even 2-dimensional system has long run period and perfect uniform distribution.

The methodology of the work could be interesting for those who deal with chaotic application to information security, chaotic PRNG design and analyses.

[1] O. Garasym, I. Taralova, *High-speed encryption method based on switched chaotic model with changeable parameters*, The international conference for internet technology and secured transactions (ICITST-2013), (2013), 41-46.

[2] R. Lozi, *Emergence of randomness from chaos*, International Journal of Bifurcation and Chaos, vol. 22, n.02 (2012), 123008 (15 pages).

COMMUTING REAL HOMOGRAPHIC FUNCTIONS

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If real homographic functions φ and ψ satisfy the inequality $\varphi \leqslant \psi$, then the difference $\varphi - \psi$ is a constant function. Basing on this observation we consider the inequality

$$\varphi \circ \psi \leqslant \psi \circ \varphi$$

in the class of homographic functions.

ON SUSPENSION FLOWS OVER COUNTABLE ALPHABET TOPOLOGICAL MARKOV CHAINS

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Both countable alphabet Markov shifts [1] and suspension flows over them [2] are powerful tools in the theory of smooth dynamics (see, e.g., [3], [4]). We first state the following general assertion giving a possibility to estimate the Kolmogorov–Sinai entropy from below in some non-compact and even non-topological situations. Let T be an automorphism of a measurable space $(X, \mathcal{F})$ and μ_n, $n = 0, 1, \ldots$, be T-invariant probability measures on $(X, \mathcal{F})$. Assume that for a countable measurable partition ξ of $(X, \mathcal{F})$, the entropy $H_{\mu_0}(\xi)$ is finite and that there exist sequences of numbers $r_n \in \mathbb{N}$ and $\varepsilon_n > 0$ such that

$$\lim_{n\to\infty} r_n = \infty, \quad \lim_{n\to\infty} \varepsilon_n = 0, \ \limsup_{n\to\infty} h_{\mu_n}(T) \geqslant h \geqslant 0,$$

$$\xi \ \text{ is a generator for } \ (T, \mu_n), \ n \geqslant 0,$$

$$|\mu_0(A) - \mu_n(A)| \leqslant \varepsilon_n \mu_n(A) \ \text{ for } \ A \in \vee_{i=0}^{r_n} T^{-i}\xi, \ n \geqslant 0.$$

Then $h_{\mu_0}(T) \geqslant h$ where $h_{\mu_0}(T)$ is the entropy of T with respect to μ_0. We apply this theorem to the suspension flow (T, f) generated by a countable alphabet topological Markov shift T defined on a sequence space X and a function f on X. The result is a family of conditions on a T-invariant probability measure $\bar{\mu}$ on X under which the (T, f)-invariant probability measure $\bar{\mu}_f$ on X_f, the phase space of the suspension flow (T, f), is a measure with maximal entropy for (T, f). (We assume that f is a function with summable variations in which case such measures $\bar{\mu}$ and $\bar{\mu}_f$ are unique provided that $\bar{\mu}$ exists.) Close conditions are stated in [3], Theorem 2.1. The only difference is that in [3] the entropy of the flow (T, f) with respect to $\bar{\mu}_f$ is assumed to be known, while now we replace this assumption by the requirement that the entropy of the partition of X into the one-dimensional cylinders corresponding to the 0th coordinate with respect to $\bar{\mu}$ is finite. It is quite possible that this requirement can be removed.

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[2] I. Cornfeld, S. Fomin, Ya. Sinai, *Ergodic theory*, Springer, NY, 1982.

[3] A. Bufetov, B. Gurevich, *Existence and uniqueness of a measure with maximal entropy for the Teichmuller flow on the moduli space of abelian differentials*, Sb. Math., **202**:7–8 (2011), 935–970.

[4] O. Sarig, *Symbolic dynamics for surface diffeomorphisms with positive entropy*, J. Amer. Math. Soc., **26**:2 (2013), 341–426.

ASYMPTOTIC STABILITY OF COCYCLES

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Let $\mathbb{T}$ be a subgroup of the additive group $(\mathbb{R}, +)$ of all reals containing as a semigroup a set $\mathbb{N}$ of all positive integers. Let $\mathbb{T}^+ := \mathbb{T} \cap (0, \infty)$. Assume that Ω (a parameter space) is a nonempty set and (X, ρ) (a phase space) is a metric space. Let $\theta = \{\theta_t : \Omega \to \Omega;\ t \in \mathbb{T}\}$ be a group of transformations. Consider the mapping $\varphi : \mathbb{T}^+ \times \Omega \to X^X$ satisfying the following equation

$$\varphi(s + t, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega)$$

for $s, t \in \mathbb{T}^+$ and $\omega \in \Omega$. A pair (θ, φ) we call a *cocycle* (over θ).

Our goal is to show some sufficient conditions for existence of an attractor of a family of multifunctions induced by cocycles which attracts globally all bounded subsets of X. We show that in the case of so–called uniform contractivity existence of an attractor is a property of a cocycle mapping itself and does not depend on properties of a parameter nor a state space. The notion of cocycle mapping is fundamental in the theory of nonautonomous/random dynamical system. We also consider asymptotic properties of families of Markov operators induced by random dynamical systems and we show the relationship between supports of its attracting measures and attractors of multifunctions induced by cocycle mappings. We use as a main tool topological limits of nets instead of common Haussdorf metric. The author was inspired by results by A. Lasota and J. Myjak (see [1] and [2]). Our results generalize those on iterated function systems obtained by them.

[1] A. Lasota, J. Myjak, *Markov operators and fractals*, Bull. Pol. Acad. Sci. Math., **45**, No. 2 (1997), 197–210.

[2] A. Lasota, J. Myjak, *Attractors of multifunctions*, Bull. Pol. Acad. Sci. Math., **48**, No. 3 (2000), 319–334.

FORMAL SOLUTIONS OF ITERATIVE FUNCTIONAL EQUATIONS

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We obtain formal series solution of the iterative functional equations

$$f^2(x) = x + e^x,\ g^2(x) = xe^x,\ h^2(x) = x^x.$$

These three equations are analytically conjugate, so their solutions have the same coefficients at different places.

Moreover, we consider iterative roots of polynomial functions.

ALGEBRAIC OPERATORS AND HAMEL BASES

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In [1, 2] discontinuous additive involutions (iterative roots of identity of order 2) on a real topological vector space X are considered. It is proved there that both, the class of such the involutions which save a Hamel base and the class of involutions which save no infinite system of vectors linearly independent over $\mathbb{Q}$, are dense in the topological vector space of all additive functions on X. This result has next been generalized for discontinuous additive iterative roots of identity of arbitrary order ([3]). We discus similar properties for algebraic operators, i.e. for additive functions $a : X \to X$ satisfying

$$p(a) = 0,$$

where $p \in \mathbb{Q}[x]$ is a polynomial.

[1] K. Baron, *On additive involutions and Hamel bases*, Aequationes Math. 2013, DOI: 10.s00010-012-0183-5.

[2] W. Jabłoński, *Additive involutions and Hamel bases*, Aequationes Math. 2013, DOI: 10.1007/s00010-013-0241-7.

[3] W. Jabłoński, *Additive iterative roots of identity and Hamel bases* (manuscript).

PARAMETRIZED MEANS AND THEIR GAUSS COMPOSITIONS

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We propose the notion of a mean depending on a parameter, define the Gauss-type iterates of parametrized mean-type mapping and study their limit behaviour. Also a suitable invariance property of the limit function is established. In such a way we obtain an extension of Generalized Gaussian Algorithm to the case of a compact set of parameters. In particular, we generalize some well known results of J. Matkowski dealing with iterates of mean-type mappings not depending on parameter.

ITERATIVE ROOTS OF PIECEWISE MONOTONIC FUNCTIONS OF NONMONOTONICITY HEIGHT NOT LESS THAN 2 REVISITED

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This is a short report on a progress in the research made with Liu Liu and Weinian Zhang from Chengdu and with Justyna Jarczyk from Zielona Góra last years. We deal with iterative roots of continuous piecewise monotonic selfmappings of a closed interval in the case when the height of the considered function is at least 2 and the order of the root is equal to the number of forts of the function. It is known that any such root can be of one of two possible types. Roots of type $\mathcal{T}_1$ have been described in [1]. In the talk results dealing with roots of type $\mathcal{T}_2$ will be presented.

[1] Liu Liu, W. Jarczyk, Lin Li, Weinian Zhang, *Iterative roots of piecewise monotonic functions of nonmonotonicity height not less than 2*, Nonlinear Anal. **75** (2012), 286-303.

ITERATES OF RANDOM-VALUED FUNCTIONS AND MARKOV OPERATORS

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Assume that $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space and X is a metric space. Given a product measurable function $f : X \times \Omega \to X$ we examine connections between the iterates $f^n : X \times \Omega^{\mathbb{N}} \to X$ (in the sense of [1]) of f and the Markov operator P with adjoint P^* of the form $P^*\varphi(x) = \int_\Omega \varphi(f(x,\omega))\mathbb{P}(d\omega)$. Moreover some results concerning the existence and the uniqueness of solutions $\varphi : X \to \mathbb{R}$ of the equation $P^*\varphi = \varphi$ will be also presented.

[1] K. Baron, M. Kuczma, *Iteration of random-valued functions on the unit interval*, Colloq. Math. **37** (1977), 263–269.

SOME PROPERTIES OF SINGULAR HYPERBOLIC AND LORENZ-TYPE ATTRACTORS

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We discuss the problem of topological distinguishing of singular hyperbolic attractors. The results obtained by author about Lorenz-like attractors and their generalizations are used.

EMBEDDABILITY OF HOMEOMORPHISMS OF THE CIRCLE IN SET-VALUED ITERATION GROUPS

Dorota Krassowska
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Let $F : S^1 \to S^1$ be a homeomorphism without periodic points. It is known that the set $L_F := L_F(z) = \{F^n(z) : n \in \mathbb{Z}, z \in S^1\}^d$, the set of limit points of orbits of F, does not depend on the choice of $z \in S^1$ and that L_F either equals S^1 or L_F is a nowhere dense perfect set. In the case when $L_F = S^1$ homeomorphism F is embedded in a continuous iteration group (uniquely with a precision up to a constant).
We deal with the case $L_F \neq S^1$. In this case F can be embedded but does not have to be, moreover, if embeddability is possible, then only in a nonmeasurable iteration group.
For a given homeomorphism without periodic points we construct some substitute of an iteration group, namely the special set-valued iteration group $\{F^t : S^1 \to cc[S^1], t \in T\}$, with restricted set of indices (time parameter) T, where T is a countable and dense subgroup of $\mathbb{R}$, which is regular in a sense and in which F can be embedded. If there exists a nonmeasurable embedding $\{f^t : S^1 \to S^1, t \in \mathbb{R}\}$ of F, then there exists an additive function $\gamma : \mathbb{R} \to T$ such that $f^t(z) \in F^{\gamma(t)}(z), t \in \mathbb{R}$. We determine a unique maximal subgroup T with this property. If $\frac{1}{n} \in T$ then F possesses a multivalued n-th iterative root.

ON STRONGLY IRREGULAR POINTS OF A BROUWER HOMEOMORPHISM EMBEDDABLE IN A FLOW

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We present properties of the set of all strongly irregular points of a Brouwer homeomorphism which is embeddable in a flow. We show that this set plays a crucial role in studying the problem of topological conjugacy for flows of Brouwer homeomorphisms. In particular, we give a sufficient condition for generalized Reeb flows to be topologically conjugate.

NON-MONOTONIC ITERATIVE ROOTS EXTENDED FROM CHARACTERISTIC INTERVALS

Liu Liu[1] and Weinian Zhang[2]
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[2]*Sichuan University, China*

Because iteration of non-monotonic one-dimensional mappings may exhibit complicated dynamics, few results on continuous iterative roots were given for non-monotonic functions except for PM functions. For some PM functions which do not increase the number of forts under iteration a method was given to obtain a non-monotonic iterative root by extending a monotone iterative root from the characteristic interval. In this paper we prove that every continuous iterative root is an extension from the characteristic interval and give various modes of extension for those iterative roots of PM functions.

[1] L. Liu, W. Zhang, *Non-monotonic iterative roots extended from characteristic intervals*, J. Math. Anal. Appl. **378** (2011), 359-373.

THE STRUCTURE OF DENDRITES ADMITTING THE EXISTENCE OF AN ARC HORSESHOE

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Let X be a dendrite or a graph, $f : X \to X$ is a continuous map. We say that f *has*
• *a horseshoe*, if there exist nonempty disjoint sub-continua $A, B \subset X$ such that $f^n(A) \cap f^n(B) \supset A \cup B$;
• *an arc horseshoe*, if there are arcs $A, B, C \subset X$ such that $A, B \subset C$ and A, B form a horseshoe for f.

In [1] one demonstrates that for a continuous map f on a graph the positivity of a topological entropy of f is equivalent to the existence of an arc horseshoe for some iteration of f. In [2,3] one constructs examples of continuous maps on dendrites with a positive topological entropy that have a horseshoe, but no iteration of the map has an arc horseshoe.

We say that *a dendrite X admits the existence of an arc horseshoe*, if for any continuous map $f : X \to X$ having a horseshoe there exists a natural number $n \geqslant 1$ such that f^n has an arc horseshoe. In the report the structure of dendrites admitting the existence of an arc horseshoe is investigated.

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HOMOGRAPHIC COMMUTING FUNCTION

Janusz Matkowski
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We consider the commuting homographic functions as mappings of the closed complex plane. We prove that every homographic function is a composition of two homographic involutions. Every homographic involution, different than identity, has exactly two fixpoints, and a homography different than the identity, is an involution iff it has a fixpoint (a cycle) of order two. Using these facts, we show that two homographic functions, not being involutory, commute, iff they have the same fixpoints. Commuting involutions are also determined.

L^P-SOLUTIONS OF INHOMOGENEOUS REFINEMENT TYPE EQUATIONS

Janusz Morawiec and Rafał Kapica
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Given measure spaces $(\Omega_1, \mathcal{A}_1, \mu_1), \ldots, (\Omega_N, \mathcal{A}_N, \mu_N)$ and transformations $\varphi_1 \colon \Omega_1 \times \mathbb{R}^m \to \mathbb{R}^m, \ldots, \varphi_N \colon \Omega_N \times \mathbb{R}^m \to \mathbb{R}^m$, $g \colon \mathbb{R}^m \to \mathbb{R}$ we discuss the problem of the existence of L^p-solutions $f \colon \mathbb{R}^m \to \mathbb{R}$ of the equation

$$f(x) = \sum_{n=1}^{N} \int_{\Omega_n} \Big| \det(\varphi_n)'_x(x,\omega) \Big| f(\varphi_n(x,\omega)) d\mu_n(\omega) + g(x).$$

EXISTENCE OF ITERATIVE ROOTS FOR A CLASS OF PIECEWISE MONOTONE CONTINUOUS FUNCTIONS

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Zhang [2] raised the following problem: "Does any piecewise monotone function on $I = [a, b]$, with $H(F) \geqslant 2$ have an iterative root of order k when the number of forts of F is less than k?" Liu et al [3] gave a necessary and sufficient condition for existence of iterative root of order $k = N(F)$. In this paper, we continue to answer the question with a natural condition $N(f) = N(F) - 1$, and established that iterative roots of order two are the only possible roots. Existence theorems of such iterative roots are given.

[1] M. Kuczma, B. Choczewski, R. Ger, *Iterative functional equations*, Encyclopedia of Mathematics and its Applications 32, Cambridge University Press, Cambridge, 1990.

[2] W. Zhang, *PM functions, their characteristic intervals and iterative roots*, Ann. Polon. Math. **65** (2) (1997) 119 - 128.

[3] L. Liu, W. Jarczyk, L. Li, W. Zhang, *Iterative roots of piecewise monotonic functions of nonmonotonicity height not less than 2*, Nonlinear Anal. **75** (1) (2012) 286 - 303.

INTERPOLATION PROBLEM IN THE KERNEL OF THE CONVOLUTION WITH THE NODES SPECIFIED IN THE CORNER

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Let $H(\mathbb{C})$ be the space of entire functions with the topology of uniform convergence on compact sets, $H^*(\mathbb{C})$ be the space conjugate to $H(\mathbb{C})$, and $P_{\mathbb{C}}$ be the space of entire functions of exponential growth. We assign a function $\varphi \in P_{\mathbb{C}}$ the functional $F \in H^*(\mathbb{C})$ such that $\widehat{F}(z) = \varphi(z)$, where $\widehat{F}(z) = \langle F_\lambda(z), e^{\lambda z}\rangle$ is the Laplace transform of F. We write the convolution operator in $H(\mathbb{C})$ in the form

$$M_\varphi[f](z) = (F_t, f(z+t)),\ f \in H(\mathbb{C}).$$

We let $Ker\, M_\varphi$ denote the kernel of the convolution operator M_φ,

$$Ker\, M_\varphi = \{f \in H(\mathbb{C}) : M_\varphi = 0\}.$$

We solved the multipoint Vallee Poussin problem in $Ker\, M_\varphi$ with nodes $\mu_j, j = 1, 2, ...,$ which are zeros of a function $\psi \in H(\mathbb{C})$ as follows: for an arbitrary sequence of complex numbers $\mu_j, j = 1, 2, ...,$ does there exist a function $y \in Ker\, M_\varphi$ such that $y(\mu_j) = a_j$ for all j = 1,2,... In [1], this problem was solved in the case where the nodes are on the real axis. Here, we essentially generalize the problem setting: we assume that the nodes are inside some angle and can therefore be complex.

It is a joint work with V.V. Napalkov.

[1] V. Napalkov, A. Nuyatov, *The multipoint da la Vallee Poussin problem for a convolution operator*, Sb. Math. **203** (2012), 224—233.

HIGH DEGENERACY BIFURCATIONS OF NON-AUTONOMOUS PERIODIC ITERATIONS

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Bifurcations of the type A_1, A_2 and A_3 in Arnold's classification (fold, cusp and swallowtail) in non-autonomous p-periodic difference equations generated by parameter depending families with p real maps, are studied. It is proved that the conditions of degeneracy, non-degeneracy and unfolding are invariant relative to cyclic order of compositions. Invariance results are essential to the proper definition of the bifurcation A_3, and lower codimension bifurcations associated, using all the possible cyclic compositions of the fiber families of maps of the p-periodic difference equation. We present the geometric behaviour of this type of bifurcation and the different cases that illustrate the transitions in the dynamics related to the unfolding of the bifurcation and two examples of this type of bifurcation occurring in solutions of period two ($p = 2$) real maps.

REMARKS ON THE QUANTUM MARKOVIAN STATES

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Let M_{d^n} be the $d^n \times d^n$ complex matrix algebra. The C^*- algebra M_{d^n} embeds isometrically into C^*- algebra $M_{d^{n+1}}$. The completion of C^*- inductive limit of the C^*- algebras M_{d^n} denotes by Ω. A state ρ is a continuous linear nonegative functional on the C^*- algebra Ω with $\rho(I) = 1$, where I is the unit of the C^*- algebra Ω. The restriction of the state ρ on the C^*- algebra M_{d^n} has the form

$$\rho(F) = Tr(F\rho_n), \rho_n = \rho_n^*, Tr(\rho_n) = 1.$$

The positive semi-definite matrices ρ_n define the state ρ uniquely. We use the set of all pairs of finite words $x_1.....x_n, y_1....y_n$ in the alphabet $\{1, ..., d\}$ to index the rows and columns of a matrices in M_{d^n}. Now we define the notion of quantum markovian state. Let

$$\rho_1(x_1, y_1) = \frac{q(x_1y_1)q^+(x_1y_1)}{\lambda},$$

$$\rho_n(x_1....x_ny_1....y_n) = \frac{q(x_1y_1)a(x_1x_2y_1y_2)....a(x_{n-1}x_ny_{n-1}y_n)q^+(x_ny_n)}{\lambda^n}, n \geqslant 2.$$

Here the matrices Q, Q^+, A are positive semidefinite matrices. Under some conditions we prove that our definition is well defined. Entangled Markov chains from [1] are special case.

It is a joint work with Zina Bezhaeva.

[1] L. Accardi, F. Fidaleo, *Entangled Markov chains*, Annali di Matematika **184** (2005), 327- 346.

SCALES OF QUASI-ARITHMETIC MEANS DETERMINED BY INVARIANCE PROPERTY

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It is well known that if $\mathcal{P}_t$ denotes a set of power means then the mapping $\mathbb{R} \ni t \mapsto \mathcal{P}_t(v) \in (\min v, \max v)$ is both 1-1 and onto for any non-constant sequence $v = (v_1, \ldots, v_n)$ of positive numbers. Shortly: the family of power means is a scale. If I is an interval and $f \colon I \to \mathbb{R}$ is a continuous, strictly monotone function then $f^{-1}(\frac{1}{n}\sum f(v_i))$ is a natural generalization of power means, so called quasi-arithmetic mean generated by f. A famous folk theorem says that the only homogeneous, quasi-arithmetic means are power means. We prove that, upon replacing the homogeneity requirement by an invariant-type axiom, one gets a family of quasi-arithmetic means building up a scale, too.

[1] P. Pasteczka, *Scales of quasi-arithmetic means determined by invariance property*, arXiv:1406.0064.

APPROXIMATE SOLUTIONS TO THE TRANSLATION EQUATION

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It is well known that $F \colon \mathbb{R} \times I \to I$, where I is a real interval, is a continuous solution to the translation equation

$$F(s, F(t, x)) = F(t + s, x)$$

if and only if there are disjoint open intervals U_λ, $\lambda \in \Lambda$, continuous function $f \colon I \to I$ such that $f \circ f = f$ and $U_\lambda \subset f(I)$ for $\lambda \in \Lambda$ and the homeomorphisms $h_\lambda \colon \mathbb{R} \to U_\lambda$, $\lambda \in \Lambda$, such that

$$F(t,x) = \begin{cases} h_\lambda(h_\lambda^{-1}(f(x)) + t), & \text{if } f(x) \in U_\lambda, \lambda \in \Lambda, t \in \mathbb{R}; \\ f(x), & \text{if } f(x) \in f(I) \setminus \bigcup_{\lambda\in\Lambda} U_\lambda, t \in \mathbb{R}. \end{cases}$$

In this talk we deal with the approximate solutions to the translation equation, we try to characterize them in a similar way.

[1] J. Chudziak, *Approximate dynamical systems on interval*, Appl. Math. Lett. **25** (2012), 352–357.

[2] Z. Moszner, B. Przebieracz, *Is the dynamical system stable?* submitted.

[3] B. Przebieracz, *On the stability of the translation equation and dynamical systems*, Nonlinear Analysis **75** (2012), 1980-1988.

STABILITY OF THE DISTRIBUTION FUNCTION FOR PIECEWISE MONOTONIC INTERVAL MAPS

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The notion of distributional chaos has been introduced by Bert Schweizer and Jaroslav Smítal. Given two points x and y, for $t > 0$ let $F_{x,y}(t)$ be the limit inferior of the relative number of times when the distance of the orbits of x and y differs at most by t, and let $U_{x,y}(t)$ be its limit superior. Then the distribution function $F(t)$ is defined as the infimum of $F_{x,y}(t)$ over all x, y with $U_{x,y}(s) = 1$ for all $s > 0$. One calls a dynamical system distributional chaotic if $F(t) < 1$ for some $t > 0$. In fact this one type of three non-equivalent types of distributional chaos.

Suppose that $T : [0,1] \to [0,1]$ is a piecewise monotonic map on the interval, this means there exists a finite partition $\mathcal{Z}$ of $[0,1]$ into finitely many pairwise disjoint open intervals satisfying $\bigcup_{Z\in\mathcal{Z}} \overline{Z} = [0,1]$ such that $T\big|_Z$ is continuous and strictly monotonic for all $Z \in \mathcal{Z}$. Note that T need not be continuous at the endpoints of the intervals of monotonicity. Because of these discontinuities it is hard to work with the original definition of the distribution function. Hence one defines an approximating distribution function G. If T is continuous with $h_{\text{top}}(T) > 0$, and B is a mixing basic set, then G and F coincide at all points where F is continuous from the right (hence they differ at most at countably many points).

Consider a mixing basic set B of a piecewise monotonic map T with $h_{\text{top}}(T) > 0$. Then the approximating distribution function G is upper semi-continuous. Hence the upper semi-continuity of F is obtained for continuous piecewise monotonic maps T. The result in the continuous case has been proved by Francisco Balibrea, Bert Schweizer, Abe Sklar and Jaroslav Smítal.

GENERALIZED ACZÉL-JABOTINSKY DIFFERENTIAL EQUATIONS AND THEIR ROLE IN ITERATION THEORY

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[1] *Karl-Franzens-University Graz, Austria*
[2] *University of Luxembourg, Grand Duchy of Luxembourg*

Let $(\Gamma, \circ)$ be the group of invertible formal power series over $\mathbb{C}$ with respect to substitution $\circ$, and let $(F_t)_{t\in\mathbb{C}}$ be an analytic iteration group in Γ. Write $H(X) := \frac{\partial F_t}{\partial t}(X)|_{t=0}$ for the generator of $(F_t)_{t\in\mathbb{C}}$. The following facts are well known:

1. The Aczél-Jabotinsky differential equation
$$H(X)\frac{\partial F_t}{\partial X}(X) = H(F_t(X)), \qquad t \in \mathbb{C} \tag{AJ,H}$$
holds. This equation can be used to construct all analytic iteration groups.

2. Equations (AJ,H) are an important tool to characterize the maximal abelian subgroups of Γ.

3. If $(F_t)_{t\in\mathbb{C}}$ and $(S^{-1} \circ F_t \circ S)_{t\in\mathbb{C}}$ are two simultaneously conjugate analytic iteration groups, then their generators H and G and S are related by the generalized Aczél-Jabotinsky differential equation
$$H(X)\frac{\partial S}{\partial X} = G(S(X)), \tag{AJ,(H,G)}$$
which is also sufficient for simultaneous conjucagy.

Let us denote by $\mathcal{L}_H$ the set of all invertible solutions of (AJ,H) and by $\mathcal{L}_{H,G}$ the set of all invertible solutions of (AJ,(H,G)) (the latter set may be empty).
In this talk we present some elementary algebraic facts about the structure of $\mathcal{L}_H$ and $\mathcal{L}_{H,G}$, and how they are related. We introduce an equivalence relation in the set of all generalized Aczél-Jabotinsky differential operators
$$\Phi \mapsto H(X)\frac{\partial \Phi}{\partial X} - G(\Phi(X)) \qquad (\Phi \in \Gamma).$$
which induces a natural equivalence relation on the set of all $\mathcal{L}_{H,G}$.

TAKAGI FUNCTION AS A UNIQUE SOLUTION OF SOME FUNCTIONAL EQUATIONS

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We prove some new characterizations of the Takagi function via functional equations and inequalities in a certain function class.

[1] H.-H. Kairies, *Takagi's function and its functional equations*, Wyż. Szkoł. Ped. Kraków. Rocznik Nauk.-Dydakt. Prace Mat. **196** (1988), 73-82.

[2] H.-H. Kairies, W.F. Darsow, M.J. Frank, *Functional equations for a function of van der Waerden type*, Rad. Mat. **4** (1988), 361-374; Errata, Rad. Mat. **5** (1980), 179-180.

[3] Pl. Kannappan, *Functional Equations and Inequalities with Applications*, Springer, 359-370, New York, 2009.

[4] J. Matkowski, *Subadditive periodic functions*, Opuscula Math. **31** (2011), 75-96.

NON-UNIQUENESS AND EXOTIC SOLUTIONS OF CONJUGACY EQUATIONS

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Conjugacy equations arise from the problem of identifying dynamical systems from the topological point of view. It is well-known that when conjugacies exist they cannot, in general, be expected to be smooth. We show that even in the simplest cases, e.g. piecewise linear maps, solutions of functional equations arising from conjugacy problems may have exotic properties: they may be singular, fractal or even everywhere discontinuous. We provide an explicit formula showing how, in certain cases, a solution can be constructively determined. We find, for the same equation, remarkably distinct solutions.

ON SOLUTIONS OF THE FUNCTIONAL EQUATIONS X(F(X(T))) = G(X(T))

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The question on the general solution of functional equations of the form

$$x(f(x(t))) \; = \; g(x(t)), \tag{1}$$

where $t \in \mathbb{R}$, $f, g : \mathbb{R} \to \mathbb{R}$, is considered [1,2]. We prove, in particular, if f is a continuous monotone map, then each interval $V \subset \mathbb{R}$ for which $g(V) \subset V$, defines a family of solutions of the equation (1); each continuous function of the form

$$x(t) = \begin{cases} gf^{-1}(t), & \text{where } t \in f(V), \\ p(t), & \text{where } t \in \mathbb{R} \setminus f(V) \text{ and } p(t) \text{ —} \\ & \text{arbitrary function with } p(t) \in V, \end{cases}$$

and only such a function is a continuous solution of equation (1).

[1] A.N. Sharkovsky, *On functional and functional-differential equations with the deviation of the argument depending on the unknown function*, in: Functional and functional-difference equations, Inst. Math., Acad. Sci. Ukrain. SSR, Kiev, 1974, 148-155 (Russian).

[2] G.P. Pelyuh, A.N. Sharkovsky, *Method of invariants in theory of functional equations*, Proceedings of Inst. Math., National Acad. Sci. of Ukraine, **95**, Kiev, 2013, 255 p. (Russian).

CHAOTIC MAPS GENERATING BIG SETS OF PROBABILITY DISTRIBUTION FUNCTIONS

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The talk is based on a joint result with *F. Balibrea* and *M. Štefánková*. For a topological dynamical system (X, f) we consider the structure of the set $\mathcal{D}(f)$ of probability distribution functions of the distances between pairs of trajectories. If f has the weak specification property then $\mathcal{D}(f)$ is closed and convex, and it can contain all nondecreasing functions $[0, \operatorname{diam}(X)] \to [0, 1]$. We show that such behavior is possible also for systems with zero topological entropy. The property is related to distributional chaos.

EMBEDDINGS OF DIFFEOMORPHISMS OF THE PLANE IN REGULAR ITERATION SEMIGROUPS

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Let $r \geqslant 1$ be an integer, $0 \leqslant \delta \leqslant 1$ and let $U \subset \mathbb{R}^2$ be a neighbourhood of 0. A diffeomorphisms $F\colon U \to \mathbb{R}^2$ such that $F(0) = 0$ is said to be of class $C^r_\delta(U)$ if it is of class $C^r(U)$ and

$$d^{(r)}F(x) = d^{(r)}F(0) + O(\|x\|^\delta), \quad \|x\| \to 0.$$

We give the full description of the $C^r_\delta(V)$, $V \subset U$ embeddings of a given diffeomorphism $F \in C^r_\delta(U)$ such that 0 is the only a hyperbolic fixed point of F. That is we determine all families of C^r_δ diffeomorphisms of the plane defined in some neighbourhood V of the origin such that $F^t \circ F^s = F^{t+s}$, $t, s \geqslant 0$, $F^1 = F$ and the mapping $t \mapsto F^t(x)$ is continuous.

CURVE SHORTENING BY SHORT RULERS

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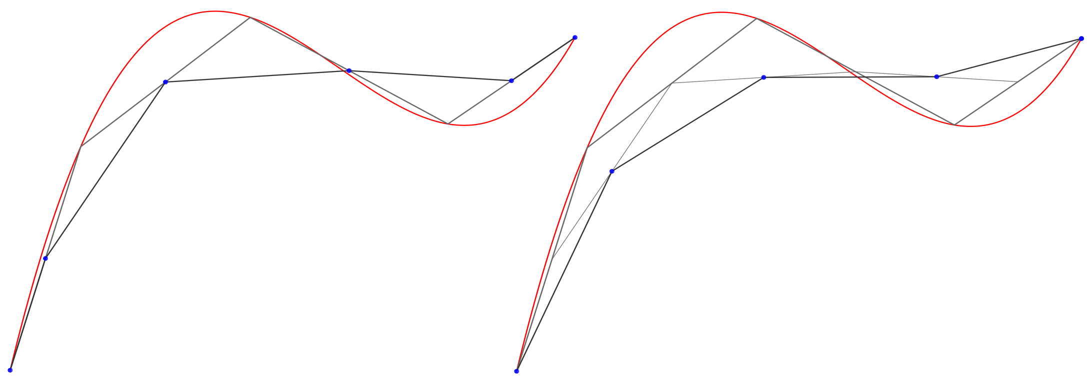
We look at homomorphisms $h\colon (\mathbb{R}, +) \to (G, \circ)$ on a Lie group G:

$$h(s+t) = h(s) \circ h(t), h(0) = e \text{ and } h(1) = g$$

The restriction of h to the interval $[0, 1]$ is a geodesic, i.e., a locally shortest line. The problem is to construct long geodesics. But any curve connecting starting point and end point can be shortened by using a ruler which allows to construct short geodesics:



In normed vector spaces, the curve converges to the straight line if it's shortened iterative. This result can be generalized to some Riemannian manifolds as the hyperbolic spaces.

INHERITING OF CHAOS IN NONAUTONOMOUS DYNAMICAL SYSTEMS

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We consider uniform nonautonomous discrete dynamical systems $\{f_n\}_{n\geqslant 1}$, where every f_n is a surjective continuous map $[0,1] \to [0,1]$ such that f_n converges uniformly to a map f. We show, among others, that if f is chaotic in the sense of Li and Yorke then the nonautonomous system $\{f_n\}_{n\geqslant 1}$ is Li-Yorke chaotic as well, and that the same is true for distributional chaos. If f has zero topological entropy then the nonautonomous system inherits its infinite ω-limit sets.

DISCUSSION ON POLYNOMIALS HAVING POLYNOMIAL ITERATIVE ROOTS

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In this work we discuss polynomial mappings which have iterative roots of polynomial form. We apply the computer algebra system *Singular* to decompose algebraic varieties and finally find a condition under which polynomial functions have quadratic iterative roots of quadratic polynomial form. This condition is equivalent to but simpler than Schweizer and Sklar's and more convenient than Strycharz-Szemberg and Szemberg's. We further find all polynomial functions which have cubic iterative roots of quadratic polynomial form and compute all those iterative roots. Moreover, we find all 2-dimensional homogeneous polynomial mappings of degree 2 which have iterative roots of polynomial form and obtain expressions of some iterative roots.

This is a joint work by Zhiheng Yu, Lu Yang and Weinian Zhang.

[1] X. Chen, Y. Shi, W. Zhang, *Planar quadratic degree-preserving maps and their iteration*, Results Math. **55** (2009), 39-63.

[2] P. Gianni, B. Trager, G. Zacharias, *Gröbner bases and primary decomposition of polynomials*, J. Symbolic Comput. **6** (1988), 146-167.

[3] B. Schweizer, A. Sklar, *Invariants and equivalence classes of polynomials under linear conjugacy*, Contributions to General Algebra **6**, Holder-Pichler-Tempsky, Vienna, 1988, 253-257.

[4] B. Strycharz-Szemberg, T. Szemberg, *Geometry of the locus of polynomials of degree 4 with iterative roots*, Centr. Europ. J. Math. **9** (2010), 338-345.

ON CONVEX ITERATION SEMIGROUPS

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Let I be a closed interval. A function $f : I \to I$ is said to be *iteratively convex* if f possesses convex iterative roots of all orders.

A family of functions $\{f^t : I \to I\}$ is said to be a *convex iteration semigroup of* f if $f^t \circ f^s = f^{t+s}$, $t, s \geqslant 0$, $f^1 = f$ and all f^t are convex. We accept the analogous definitions for the concave functions.

If f is of class C^1 and $f'(p) > 0$ for a fixed point p of f, then f is iteratively convex if and only if f possesses a convex iteration semigroup.

The problem of description of iterative convexity can be reduced to the functions satisfying the condition

(H) $f : [0,1] \to [0,1]$ is strictly increasing of class C^1 and $0 < f(x) < x$ for $x \in (0,1)$.

Further, let f satisfy (H) and $f'(0) \neq 0$. If f and f' are concave then f is iteratively concave. For convex functions the situation is more complicate.

If f of class C^3 is convex and $f''(0) \neq 0$ then f is iteratively convex in a neighbourhood of 0. If f is iteratively convex on $[0, \delta]$ then for every $\delta < a < 1$ there exists a unique iteratively convex extension $\widetilde{f}$ on $[0,1]$ such that $\widetilde{f}_{[a,b]}$ is affine.

Let $0 < \lambda_0 < 1 < \lambda_1$ and denote by $Conv(\lambda_0, \lambda_1)$ the set of all convex functions f satisfying (H) such that $f(1) = 1$, $f'(0) = \lambda_0$ and $f'(1) = \lambda_1$. The set of all iteratively convex functions is of the first category in the space $Conv(\lambda_0, \lambda_1)$ endowed with the classical metric in $C^2[0,1]$ space.

The iterative convexity is not stable. If $f \in Conv(\lambda_0, \lambda_1)$ is iteratively convex then for every $p \in (0,1)$ there exists a neighbourhood U of p such that every convex function F for which $F|_{[0,1]\setminus U} = f|_{[0,1]\setminus U}$ and $F|_U > f|_U$ is not iteratively convex.

SOME ADVANCES ON LINEARIZATION

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C^1 linearization is of special interests because it can distinguish characteristic directions of dynamical systems. It is known that planar $C^{1,\alpha}$ contractions with a fixed point at the origin admit $C^{1,\beta}$ linearization with sufficiently small $\beta > 0$ if $\alpha = 1$ and admit $C^{1,\alpha}$ linearization if $(\log|\lambda_1|/\log|\lambda_2|) - 1 < \alpha \leqslant 1$, where λ_1 and λ_2 are eigenvalues of the linear parts of the contractions with $0 < |\lambda_1| \leqslant |\lambda_2| < 1$. In this talk we present some advances that the lower bound of α is improved and the obtained Hölder exponent β of the linearization is sharp.

This results are obtained jointly by Wenmeng Zhang, Witold Jarczyk and Weinian Zhang.

[1] G.R. Belitskii, *Equivalence and normal forms of germs of smooth mappings*, Russian Math. Surv. **33** (1978), 107-177.

[2] P. Hartman, *Ordinary differential equations*, Wiley, New York, 1964.

[3] K. Lu, *A Hartman-Grobman theorem for scalar reaction-diffusion equations*, J. Differential Eq. **93** (1991), 364-394.

[4] H.M. Rodrigues, J. Sol-Morales, *Linearization of class C^1 for contractions on Banach spaces*, J. Differential Eq. **201** (2004), 351-382.

[5] G.R. Sell, *Smooth linearization near a fixed point*, Am. J. Math. **107** (1985), 1035-1091.

[6] D. Stowe, *Linearization in two dimensions*, J. Differential Eq. **63** (1986), 183-226.

[7] W.M. Zhang, W.N. Zhang, *C^1 linearization for planar contractions*, J. Functional Anal. **260** (2011), 2043-2063.

[8] W.M. Zhang, W.N. Zhang, W. Jarczyk, *Sharp regularity of linearization for $C^{1,1}$ hyperbolic diffeomorphisms*, Math. Ann. **358** (2014), 69-113.